

Reply to M. Campisi [arXiv: 1310.5556]

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In response to M. Campisi's comment [arXiv: 1310.5556] on our recent work [Phys. Rev. E 88, 042126 (2013)], we first point out that the distribution used by Campisi is not the correct escort distribution and further provide arguments showing that the distributions obtained from the finite bath scenario are not Tsallis distributions assuming the ergodicity of the total system. We also comment on the role of evidence mentioned by M. Campisi.

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Before proceeding in our reply to the comment by M. Campisi [1] on our recent work [2], we first define the Tsallis entropy [3]

$$S_q = \frac{\int p^q d\Gamma - 1}{1 - q} \quad (1)$$

without the multiplicative constant k , where p is the probability distribution, q is the nonextensivity parameter, and Γ denotes the phase space variable. If one maximizes the entropy above with the usual normalization condition $\int p d\Gamma = 1$ and the escort internal energy constraint $\frac{\int p^q H_S d\Gamma}{\int p^q d\Gamma} = U$ where H_S is the system Hamiltonian and U is the average energy, one obtains the following Tsallis distribution

$$p = \left[1 - (1 - q) \frac{\beta}{\int p^q d\Gamma} (H_S - U) \right]^{\frac{1}{1-q}}. \quad (2)$$

where β is the Lagrange multiplier associated with the internal energy constraint [3]. The reader can check this by inspecting Eqs. (3.197) and (3.198) in Ref. [3]. Note that the normalization constant is suppressed from here on, since it is irrelevant to the present discussion.

Campisi, instead of Eq. (2) above, prefers to raise the exponent of the distribution in Eq. (2) to the power of q and defining $P_{\text{Campisi}} = p^q$, writes

$$P_{\text{Campisi}} = \left[1 - (1 - q) \beta (H_S - U) \right]^{\frac{q}{1-q}} \quad (3)$$

as can be seen from Eq. (4) and the footnote given under the heading of Reference 6 in his comment. However, a quick comparison of Eqs. (2) and (3) reveals that for $P_{\text{Campisi}} = p^q$ to be correct, one must also have $\int p^q d\Gamma = 1$ for $\frac{\beta}{\int p^q d\Gamma}$ to be equal to β . If this assumption of Campisi i.e. $\int p^q d\Gamma = 1$ is however indeed realized, the equilibrium Tsallis entropy, irrespective of the physical system under scrutiny, always yields zero, which can be seen by substituting $\int p^q d\Gamma = 1$ into Eq. (1).

It is also important to note that even one grants Campisi the condition $P_{\text{Campisi}} = p^q$ neglecting the above discussion (as a result of using a different constraint in obtaining p for example), P_{Campisi} , being an equilibrium distribution, should be normalized i.e., $\int P_{\text{Campisi}} d\Gamma = 1$ i.e., $\int p^q d\Gamma = 1$. This normalization again, substituted in the Tsallis

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entropy given by Eq. (1), leads to the same anomaly of assigning zero entropy value for all systems at equilibrium which is not correct.

Concerning the equipartition theorem and finite baths, both us and Campisi agree on the fact that the equipartition theorem (see Eq. (5) in the comment by Campisi) should be intact in nonextensive scenario i.e. the physical inverse temperature β (and the physical temperature itself for that matter) must be independent of q . This is tantamount to the fact that the relevant term appearing in the Tsallis distribution should simply be β as is also required by the finite bath scenario. The inspection of the correct escort distribution in Eq. (2) shows that this is not so i.e. instead of β , we have a term $\frac{\beta}{\int p^q d\Gamma}$ which is q -dependent. We might choose to consider this term as the physical inverse temperature, but this happens at the expense of violating the equipartition theorem (which simply states that the temperature must be q -independent) as we noted in our work [2]. Other option is to set $\int p^q d\Gamma = 1$ in Eq. (2). But, as we explained above, the Tsallis entropy given by Eq. (1) then becomes zero for any system at equilibrium which is nonsensical.

When a physical system is in weak contact with a thermal bath of finite constant heat capacity C_B , the probability distribution of the system reads

$$\rho = (E_{tot} - H_S)^{C_B - 1}, \quad C_B > 0, \quad (4)$$

where E_{tot} is the total energy of the system and the bath. When the bath has negative finite heat capacity i.e. $C_B < 0$, albeit constant again, the system distribution possesses the same form, but the term $(E_{tot} - H_S)$ is replaced by $(H_S - E_{tot})$ as Campisi notes in Eqs. (1) and (2) in his comment. To be able to produce the Tsallis distributions from these system distributions, as Campisi notes (see below Eq. (3) in his comment), one has to use the relation

$$C_B = \frac{1}{1 - q} \quad (5)$$

for all values of q . The system probability distributions given by Eq. (4) (and its counterpart for the case $C_B < 0$) have to satisfy two important physical limits, namely, the limits $C_B \rightarrow \infty$ and $C_B \rightarrow 0$. The former limit corresponds to the well-known canonical distribution when the heat capacity of the bath becomes infinite whereas the latter yields the microcanonical ensemble where the system is completely isolated. These limits are also mathematically important, since the limit $C_B \rightarrow 0$ for example must exist due to the continuity of the finite baths formalism. Note that both these limits are also agreed on by Campisi [5]. These limits can easily be taken to produce the canonical distribution and the microcanonical Dirac delta as can be easily verified by using Eq. (4) and its $C_B < 0$ counterpart.

On the other hand, if the relation in Eq. (5) should make any sense within the Tsallis formalism, these same limits must be taken in terms of the q values in accordance with Eq. (5). The limiting behavior $C_B \rightarrow \infty$, due to Eq. (5), corresponds to the limit $q \rightarrow 1$ of the escort distribution in Eq. (2), yielding indeed to the canonical distribution in nonextensive thermostatics.

However, the second limiting behavior $C_B \rightarrow 0$ corresponding to the microcanonical ensemble is problematic for the Tsallis distributions in Eq. (2) although this limit can easily be taken to obtain the Dirac delta distribution through the finite bath system distributions given by Eq. (4) and its counterpart for $C_B < 0$. The inspection of Eq. (5) shows that this limiting behavior corresponds to the limit $q \rightarrow \pm\infty$, depending on whether one approaches from the right $C_B \rightarrow 0^+$ (i.e. $q \rightarrow -\infty$) or left $C_B \rightarrow 0^-$ (i.e. $q \rightarrow +\infty$). The limits $q \rightarrow \pm\infty$ cannot be realized in the Tsallis escort distributions given by Eq. (2): The first limit i.e. $q \rightarrow -\infty$ cannot be realized, since the Tsallis entropy given by Eq. (1) is concave only for $q > 0$, which can easily be seen from the condition $\frac{\partial^2 S_q}{\partial p^2} < 0$ i.e.

$$\frac{\partial^2 S_q}{\partial p^2} < 0 \quad \implies \quad \frac{q(q-1)p^{q-2}}{(1-q)} = -qp^{q-2} < 0 \quad \implies \quad q > 0. \quad (6)$$

This implies that even if one mathematically forces the limit $q \rightarrow -\infty$ on the Tsallis distributions and obtains a microcanonical distribution, one will never know that this is the distribution maximizing the Tsallis entropy.

Concerning the limit $q \rightarrow +\infty$, the permissible q -values are limited within the range $1 < q < 1 + \frac{2}{dN_S}$ where N_S and d denote the number of particles in the system and the dimensionality, respectively as Lutsko and Boon [4] first showed, and as we confirmed (see the paragraph below Eq. (17) in our work [2]). Due to this permissible interval of q values, the limit $q \rightarrow +\infty$ requires either d , or the number of particles in the system N_S to attain a value of zero which is nonsensical whereas the limit $C_B \rightarrow 0$, using the system distribution in Eq. (4) and its $C_B < 0$ counterpart without relating it to the parameter q , is permissible and unproblematic. We also note that the above argument is applicable to any kind of Tsallis distributions, independently of the constraints, since concavity is a property of

the Tsallis entropy itself, and the Tsallis distributions cannot attain the limit $q \rightarrow +\infty$ due to the normalization requirement.

The simulation Campisi mentions in the fourth paragraph of his comment is not in contradiction with our results, but instead supports them: since there is neither a single occurrence of the nonextensivity parameter q nor Tsallis distributions themselves in the aforementioned simulations and related calculations in Ref. [5], the work by Campisi *et al.* [5] simply shows that the finite bath distributions can be studied in its full generality without any additional nonextensive ingredient.

In sum, the equilibrium distributions originating from the finite baths, assuming the ergodicity of the total compound, are not Tsallis distributions. However, our results do not exclude the possibility of Tsallis distributions in non-ergodic systems.

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